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The Relation Between Boundedness and Oscillatory of First Order Neutral Delay Differential Equation

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ABSTRACT

It is well known that all types of differential equations have wide range of applications in many areas. there is special type for differential equations that play important different theories of mechanical and electrical systems such by renaming the variables, this analogy has an important application the second order nonlinear differential equations with variables coefficients is of special interest, in this work we consider and give some sufficient conditions for oscillation of solutions of some types of neutral delay differential equations. and we made connection between oscillations and bondedness of the solution of the first order neutral delay differential equation of this form:

$$\left(x(t) + a(t)x(t-\tau)\right)^{\dot{}} + p(t)f\big(x(t-\alpha)\big) + q(t)g\big(x(t-\beta)\big) = 0$$

Where we will prove any oscillation solution of a differential equation is always bounded and the converse is not true. And we use MATLAB software to solve examples on the previous equation.

الخلاصة

لقد تم في هذا البحث دراسة تذبذب ومحدودية الحلول لبعض انواع المعادلات التفاضلية ذات التأخير المحايد. وحاولنا ايجاد علاقة بين التذبذب ومحدودية حلول المعادلات التفاضلية ذات التأخير المحايد من الرتبة الاولى التي على الصورة:

$$\left(x(t) + a(t)x(t-\tau)\right)^{\dot{}} + p(t)f\big(x(t-\alpha)\big) + q(t)g\big(x(t-\beta)\big) = 0$$

حيث أثبتنا أن أي حل تذبذبي للمعادلة التفاضلية يكون دائمًا محدودًا، والعكس ليس صحيحًا. وقدمنا بعض امثلة التي أستخدمنا برنامج الماتلاب لأثبات انها محدودة.

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section1: Introduction.

The delay equation is a differential equation in which the unknown function appears with delay.

A neutral delay differential equation (NDDE) is a differential equation in which the higher order derivative of the unknown function appears in the equation both with and without delays.

Example (1.1): The equation $(x(t) + px(t-\tau))' + qx(t-\sigma) = 0 \quad t \geq t_0$

where $\tau, q \in (0,\infty)$, $\sigma \in [0,\infty)$ and $p \in \Re - \{0\}$

is an example of first order neutral delay

A point $t = t_0$ is called a zero of the solution $x(t)$ if $x(t_0) = 0$

The equation is called oscillatory if it has arbitrarily large zeros and called non-oscillatory if it is eventually positive or eventually negative.

The equation is oscillatory if every solution is oscillatory

Example (1.2): Consider the equation:

$$\acute{\acute{y}}(t) + \frac{1}{2}\acute{y}(t) - \frac{1}{2}y(t-\pi) = 0 \quad \text{for } t \geq 0$$

whose solution y(t)=$1 - \sin t$ has an infinite sequence of multiple zeros.

This solution has an oscillatory property.

Example (1.3): Consider the equation:

$$\acute{\acute{y}}(t) - y(-t) = 0.$$

This equation has an oscillatory solution $y_1 = \sin t$ and a non-oscillatory solution $y_2 = e^t + e^{-t}$. Then it is called non- oscillatory.

Section2: Oscillation of natural delay differential equation.

In this section we shall study the oscillatory behavior of all solutions of neutral delay differential equations with variable coefficient and we shall establish and prove some results.

Theorem (2.1): [14] Consider the NDDE

$$(x(t) + px(t-\tau)) - q(t)x(t+\sigma) = 0, t \geq t_0 \quad \text{.......... (2.1)}$$

Where p, τ and $\sigma \in (0,\infty)$, and q (t)$\in C[[t_0,\infty), \mathbb{R}^+]$is periodic with period τ or non-decreasing function on $[t_0,\infty)$ and satisfies the condition.

$$\lim_{t\to\infty} \inf \int_{t-\sigma}^{t} q(s)ds > \frac{1+p}{e} \quad \text{.......... (2.2)}$$

Then every solution of the equation (2.1) oscillates.

Example (2.2): [14] Consider the NDDE

$$(x(t) + x(t-\tau))' - x(t+\frac{\pi}{2}) = 0, \tau \in (0,\infty), t \geq 0$$

We note that

$$p = 1, \tau \in (0,\infty), \sigma = \frac{\pi}{2} \; and \; q(t) = 1.$$

Then we have

1. p, τ and $\sigma \in (0, \infty)$.
2. $q(t) \in c[[t_0, \infty), \mathbb{R}^+]$ is periodic with period τ or non-decreasing on $[t_0, \infty)$ and

$$\liminf_{t\to\infty} \int_{t-\frac{\pi}{2}}^{t} q(s)ds = \liminf_{t\to\infty} \frac{\pi}{2} = \frac{\pi}{2} > \frac{2}{e}.$$

It follows from Theorem (2.1) that every solution of the given equation is oscillatory. Hence the given equation is oscillatory.

Theorem (2.3): [10] Consider the NDDE (2.1) were

1. p, τ and $\sigma \in (0, \infty)$.
2. $q(t) \in c[[t_0, \infty), \mathbb{R}^+]$ and satisfies the condition.

$$\liminf_{t\to\infty} \int_{t-(\sigma-\tau)}^{t} \frac{q^2(s)}{q(s)+pq(s-\tau)} ds > \frac{1}{e} \text{.................(2.3)}$$

Then every solution of the equation (2.1) oscillates.

Example (2.4): [10] Consider the NDDE

$$(x(t) + \frac{1}{2}x(t-\tau))^{\dot{}} - 3x(t+2) = 0\,, \tau \in (0\,,\infty), \quad t \geq 0$$

We note that $p = \frac{1}{2}\,, \tau \in (0\,,\infty)\,,$ and $\sigma = 2\,, q(t) = 3\,.$

then we have

(1) p, τ and $\sigma \in (0, \infty)$

(2) q(t)$= 3 \in C[[0, \infty), \mathbb{R}^+]$

and

$$\liminf_{t\to\infty} \int_{t-(\sigma-r)}^{t} \frac{q^2(s)}{q(s) + pq(s-r)} ds = \liminf_{t\to\infty} \int_{t-2}^{t} \frac{9}{3+\frac{3}{2}} ds$$

$$= 2 \liminf_{t\to\infty} \int_{t-2}^{t} ds$$

$$= 4 > \frac{1}{e}$$

It follows from Theorem (2.3) that every solution of the given equation is oscillatory. Hence the given equation is oscillatory.

Theorem (2.5): [10] Consider the NDDE

$$((t) + px(t-\tau))^{\dot{}} - (t)x(t-\sigma) = 0, \quad t \geq t_0 \text{ ....(2.4)}$$

Where

(1) p, τ and $\sigma \in (0, \infty)$ and $\sigma > \tau$

(2) q(t)$\in C[[t_0, \infty), \mathbb{R}^+]$ is non-increasing on $[t_0, \infty)$ and satisfies that

$$\liminf_{t\to\infty} \int_{t-(\sigma-\tau)}^{t} q(s+\tau)ds > \frac{1+p}{e} \text{ ....(2.5)}$$

Then every solution of the equation (2.4) oscillates. Hence the NDDE (2.4) is oscillatory.

Example (2.6): [10] Consider the NDDE

$$(x(t) + \frac{1}{3}x(t-1))^{\dot{}} - 3x(t-2) = 0, \qquad t \geq 0$$

We note that

$$p = \frac{1}{3}, \tau = 1, \quad \sigma = 2, \text{and } q(t) = 3.$$

Then we have

(1) p ,τ and $\sigma \in (0, \infty)$ with $\sigma > \tau$

(2) q(t)$\in C[[0, \infty), \mathbb{R}^+]$ is non- decreasing on $[t_0, \infty)$

and

$$\liminf_{t\to\infty} \int_{t+r-\sigma}^{t} q(s+\tau)ds = 3 \liminf_{t\to\infty} \int_{t+1-2}^{t} ds = 3 > \frac{4}{3e},$$

It follows from Theorem (2.5) that every solution of the given equation is oscillatory. Hence the given equation is oscillatory.

Theorem (2.7): [10] Consider the NDDE (2.4) where

(1) p ,τ and $\sigma \in (0, \infty)$ and $\sigma > \tau$

(2) q(t)$\in C[[t_0, \infty), \mathbb{R}^+]$ is periodic with period τ and satisfies that

$$\liminf_{t\to\infty} \int_{t-\sigma+\tau}^{t} q(s)ds > \frac{1+p}{e}$$

Then every solution of the equation (2.4) oscillates. And the equation (2.4) is oscillatory.

Section ***3***: **The relation between boundedness and oscillatory of first Order Neutral Delay differential equation.**

In this section we will try to find a way to make connection between the concept of boundedness and the oscillatory of the solution of the first Order Neutral Delay differential equations of this form

$$(x(t) + a(t)x(t-\tau)) + p(t)f(x(t-\alpha)) + q(t)g(x(t-\beta)) = 0, t \geq t_0 \text{........(3.1)}$$

Where $\tau > 0, \alpha > 0, \beta > 0, a, p, q \in C([t_0,\infty), \mathbb{R})$ and $f, q \in C(t_0, \infty), \mathbb{R})$,

Theorem (3.1): Any oscillation solution of a differential equation is always bounded, and the converse is not true.

Proof: Suppose that $x(t)$ is oscillatory, then it means that has infinite number of Zeros, which that means the solution is alternating through the x-axis,

hence $x(t)$ is bounded i.e., $x(t) \leq M$,$M > 0$.and the convers is not true

because

$$y_1 = sinx$$
$$y_2 = cosx$$

are solutions of $\acute{\acute{y}} + y = 0$ on the interval $0 \leq x \leq \pi$ which are bounded not oscillatory. we see from example the equation is bounded but it is not oscillatory.

we assume that $\mathbb{R}$ and $\mathbb{R}^+$denote the sets of all real numbers and nonnegative numbers, respectively, $\gamma = t_0 - \max\{\tau, \alpha, \beta\}$, $C([\gamma,+\infty),\mathbb{R})$ denotes the Banach space of all continuous and bounded functions on $[\gamma,+\infty)$ with norm

$\|X\| = sup_{t\geq\gamma}|X(t)|$ for all $x \in C([\gamma,+\infty),\mathbb{R})$,

$$A(N,M) = \{x \in C([\gamma,+\infty),\mathbb{R}: N \leq (t) \leq M, t \geq \gamma\}$$

where M, N$\in \mathbb{R}$ with M >N.

Theorem (3.2): [12] Assume that there exist constants $M, N < a_0$and D satisfying

1. $max\{|f(u) - f(v)|, |g(u) - g(v)|\} \leq D|u - v|, u, v \in [N, M]$.
2. $2\int_{t_0}^{+\infty} max\{|p(s)|, |q(s)|\}ds < +\infty$.
3. $0 \leq a(t) \leq a_0 < 1, t \geq t_0$.
4. $0 < N < (1 - a_0)M$.

Then equation (3.1) has a bounded non-oscillatory solution in A(M, N)

Example (3.1): Consider the first order neutral delay neutral delay differential equation:

$$\frac{d}{dt}\left[x(t) + \frac{2t^3}{1+8t^3}x(t-\tau)\right] + \frac{(t - sin(t^2 ln(1+t^2)))x^2(t-1)}{1+t^8} + \frac{(2 - t^4 - 4t^5)x^3(t-1)}{(1+t^4)^2} = 0, \quad t \geq 1$$

where τ is a positive constant. Let $t_0 = 1, \alpha = 1, \beta = 3,$

$\gamma = 1 - max\{1,3,\tau\}$

$M = 4, N = 1, a_0 = \frac{1}{4}, D = 48$ and

$$a(t) = \frac{2t^3}{1+8t^3}, p(t) = \frac{(t - sin(t^2 ln(1+t^2)))}{1+t^8}, q(t) = \frac{(2 - 3t^4 - 4t^{5})}{(1+t^4)^2}$$

$$f(u) = u^2, g(u) = u^3, (t,u) \in [t_0,+\infty) \times \mathbb{R}.$$

1-We choose $M = 4, N = 1$ and $u = 2, v = 3$ because they satisfying the conditions of theorem (3.1), and hence we have from the same theorem that

$$\begin{aligned} &max\{|f(u) - f(v)|, |g(u) - g(v)|\} \\ &= max\{|4 - 9|, |8 - 27|\} \\ &= mx\{|-5|, |-19|\} \\ &= max\{5,19\} \\ &= 19 < 48|3 - 2| \\ &19 < 48 \end{aligned}$$

2- $\int_1^{+\infty} max\{\left|\frac{s - sin(s^2 ln(1+s^2)))}{1+s^8}\right|, \left|\frac{(2-3s^4-4s^5)}{(1+s^4)^2}\right|\}ds < \infty$

MATLAB software is used to solve the previous equation

```
clear all
clc

fun= @(t) abs((t-sin((t.^2).*(log(1+t.^2)))./(1+t.^8))+ abs(((2-
(3*t.^4)-(4*t.^5))./(1+t.^4).^(2)));

q = integral(fun,1,inf);

q= 1.7738;
```

3- $0 < a(t) < a_0 < 1 \ ,t \geq 1$

$$\frac{2t^3}{1+8t^3} < \frac{1}{4} < 1$$

4- $0 < 1 < (\frac{3}{4})4$, $0 < 1 < 3$.

Then the equation has a bounded non-oscillatory solution.

Theorem (3.3): [12] Assume that there exist constants $M, N < a_0$ and D satisfying

1. $max\{|f(u) - f(v)|, |g(u) - g(v)|\} \leq D|u - v|, u, v \in [N, M]$.
2. $\int_{t_0}^{+\infty} max\{|p(s)|, |q(s)|\}ds < +\infty$.
3. $|a(t)| \leq a_0, t \geq t_0$.
4. $0 < N < (1 - 2a_0)M, a_0 < \frac{1}{2}$

Then equation (3.1) has a bounded non-oscillatory solution in $A(N, M)$.

Example (3.2):

Consider the first order neutral delay neutral delay differential equation:

$$\frac{d}{dt}\left[x(t) + \frac{t^3 cost^5}{1+8t^3}x(t-\tau)\right] + \frac{(t - cosx^5(t-3)}{1 + t^3(ln(1+t^2))} - \frac{(t^2 - t + 1)sin^3(x^2(t-4))}{1 + t^5(1+t^2)} = 0 \ , \quad t \geq 2$$

where τ is a positive constant. Let $t_0 = 2, \alpha = 3, \beta = 4,$

$M = 9, \ N = 2, \ a_0 = \frac{1}{3} \ , \ D = 30$ and

$$a(t) = \frac{t^3 cost^5}{1+3t^3} \ , p(t) = \frac{(1+t)}{1 + t^3(ln(1+t^2))}, q(t) = \frac{(t^2 - t + 1)}{1 + t^5(1+t^2)}$$

$$f(u) = cos^5(u^2), g(u) = sin^3(u^2), (t, u) \in [t_0, +\infty) \times \mathbb{R}.$$

1- We choose $M = 9, N = 2$ and $u = 3, v = 5$ because they satisfying the conditions of theorem (3.1), and hence we have from the same theorem that

$$max\{|cos^5(9) - cos^5(25)|, |sin^3(9) - sin^3(25)|\}$$
$$= max\{|0.328|, |-0.071|\}$$
$$= 0.257 \leq 30|3 - 5| \leq 30|-2| \leq 60$$
$$19 < 48$$

2- $\int_0^{+\infty}\{\left|\frac{1+s}{1+s^3(ln(1+s^2))}\right|, \left|\frac{(s^2-s+1)}{1+s^5(1+s^2)}\right|\}ds < \infty$

MATLAB software is used again to solve the previous equation

```
clear all
clc

fun=@(t)abs((1+t)./(1+(t.^3).*log(1+t.^2)))+abs((t.^2t+1)./(1+(t.^5).
*(1+t.^2)));
q = integral(fun,5,inf);

q = 0.0476;
```

3- $0 < N < (1-2a_0)M$

$$a_0 < \frac{1}{3} < \frac{1}{2}$$

$$0 < 2(1-\frac{2}{3})9$$

$$0 < 2\frac{9}{3}$$

$$0 < 2 < 3$$

4) $|a(t)| \leq a_0 \qquad t \geq 2$

$$\left|\frac{t^3 cos(t^5)}{1+3t^3}\right| \leq \frac{1}{3}$$

Then the equation has a bounded non-oscillatory solution.

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